

Comments on turbulence theory by Qian and by Edwards and McComb

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Abstract

We reexamine Liouville equation based turbulence theories proposed by Qian [Phys. Fluids **26**, 2098 (1983)] and Edwards and McComb [J. Phys. A: Math. Gen. **2**, 157 (1969)], which are compatible with Kolmogorov spectrum. These theories obtained identical equation for spectral density $q(k)$ and different results for damping coefficient. Qian proposed variational approach and Edwards and McComb proposed maximal entropy principle to obtain equation for the damping coefficient. We show that assumptions used in these theories to obtain damping coefficient correspond to unphysical conditions.

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I. INTRODUCTION

Edwards [2] proposed turbulence theory based on Liouville equation for joint probability distribution function of Fourier modes $u_\alpha(\mathbf{k}, t)$ of velocity field governed by forced Navier-Stokes equation. Following Edwards, a few more turbulence theories [3, 4, 10] were proposed to solve Liouville equation and were reviewed by Leslie [7] and McComb [8]. Edwards theory seeks Fokker-Planck model representation for Liouville equation and obtains closed set of equations for spectral density $q(k)$ and damping coefficient (total viscosity) $\omega(k)$ for stationary, isotropic turbulence. The theory failed to be consistent with Kolmogorov spectrum [1, 5] and the failure was attributed to equation for $\omega(k)$ [8]. As a modification to Edwards' theory, Edwards and McComb [3] proposed principle of maximal entropy to derive an equation for $\omega(k)$ compatible with Kolmogorov spectrum. Within the Fokker-Planck framework of Edwards, Qian [10] proposed variational approach and obtained different equation for damping coefficient consistent with Kolmogorov spectrum. Instead of using $u_\alpha(\mathbf{k}, t)$, Qian used real dynamical modal variables X_i and their governing equations which were suggested by Kraichnan [6] and later utilized by Herring [4] for his self-consistent turbulence theory. Herring's theory also uses Liouville equation and obtains equation for $q(k)$ identical to equation obtained by Edwards. In this paper, we reexamine Kolmogorov spectrum compatible theories proposed by Qian [10] and Edwards and McComb [3] to obtain damping coefficient. We show in the next two sections that assumptions made in these theories correspond to unphysical conditions.

II. VARIATIONAL APPROACH BY QIAN

For discussion purpose, hereafter we refer to Qian's variational approach as Q83. We use (Q83; #) to represent equation number (#) in Q83 paper [10]. Qian based his theory on governing equation for real dynamical modal variables X_i for stationary, homogeneous, isotropic turbulence, written as

$$\frac{dX_i}{dt} = -(\nu_i - \nu'_i)X_i + \sum_{j,m} A_{ijm}X_jX_m, \quad (\text{Q83; 8}) \quad (1)$$

where $\nu'_i X_i$ represents external driving force. Einstein summation convention of repeated indices is not utilized in Eq. (1) and in this section. It should be noted that the real dynamical

modal variables and their equations were first suggested by Kraichnan within the context of hydromagnetic turbulence [6]. Qian's theory [10] seeks Langevin model representation for isotropic turbulence in the form

$$\frac{d}{dt}X_i \simeq -\eta_i X_i + f_i, \quad \eta_i = \zeta_i + (\nu - \nu'_i) \quad (\text{Q83;12}) \quad (2)$$

by using

$$\sum_{j,m} A_{ijm} X_j X_m \cong -\zeta_i X_i + f_i \quad (\text{Q83;11}) \quad (3)$$

where $-\zeta_i X_i$ is dynamical damping term and f_i is white noise type forcing term. Qian proposed variational approach to obtain damping coefficient η_i . The approach yields equation for η_i by minimizing a function $I(\eta_i)$, written as

$$I = \sum_i \left\langle \left(\sum_{j,m} A_{ijm} X_j X_m - (-\zeta_i X_i) \right)^2 \right\rangle \quad (\text{Q83;29}) \quad (4)$$

and using

$$\frac{\partial I}{\partial \eta_i} = 0. \quad (\text{Q83;28}) \quad (5)$$

Here $\langle \rangle$ represents ensemble average. Qian considered variation in I under constraint $\phi_i = \text{constant}$, where ϕ_i is related to $\langle X_i^2 \rangle$ by

$$\langle X_i^2 \rangle = \phi_i \left(1 - \frac{\nu_i - \nu'_i}{\eta_i} \right). \quad (\text{Q83;25}) \quad (6)$$

Also, ϕ_i is proportional to spectral density $q(k)$ [10].

We now show that for $I = I(\eta_i)$ and under the constraint of $\phi_i = \text{constant}$,

$$\frac{\partial I}{\partial \eta_i} \neq 0 \quad (7)$$

within the framework of Langevin model considered by Qian. Consequently, the use of Eq. (5) to obtain η_i is in error. For stationary turbulence, solution of Eq. (2) suggests

$$2\eta_i \langle X_i^2 \rangle = F_i \quad (8)$$

in which correlation of white noise forcing term

$$\langle f_i(t) f_i(t') \rangle = F_i \delta(t - t') \quad (9)$$

is utilized. Here $\delta(t - t')$ is Dirac delta function. Using Eqs. (3), (4) and (9), function $I(\eta_i)$ can be written as

$$I = \sum_i \left\langle \left(\sum_{j,m} A_{ijm} X_j X_m - (-\zeta_i X_i) \right)^2 \right\rangle = \sum_i \langle f_i f_i \rangle = \delta(0) \sum_i F_i. \quad (10)$$

Further, using Eqs (6), (8), (10) and for $\phi_i = \text{constant}$, we can write

$$\frac{\partial I}{\partial \eta_i} = \delta(0) \sum_j \frac{\partial F_j}{\partial \eta_i} = \delta(0) \sum_j \frac{\partial 2\eta_j \langle X_j^2 \rangle}{\partial \eta_i} = 2\delta(0)\phi_i. \quad (11)$$

This Eq. (11) suggests that for all i

$$\frac{\partial I}{\partial \eta_i} \neq 0 \quad (12)$$

as $\phi_i \neq 0$. In view of this, use of Eq. (5), i.e. $\frac{\partial I}{\partial \eta_i} = 0$, in Q83 to obtain η_i is in error and corresponds to unphysical condition $\phi_i = 0, \forall i$ for stationary turbulence.

Now we suggest possible modification within the framework of Q83. Consider a function V , written as

$$V = \sum_j \frac{\partial I}{\partial \eta_j} = \sum_j 2\delta(0)\phi_j, \quad (13)$$

which satisfies an exact condition

$$\frac{\partial V}{\partial \eta_i} = 0. \quad (14)$$

This condition along with Eqs. (4) can be used, instead of Eq. (5), to obtain equation for η_i .

III. MAXIMAL ENTROPY PRINCIPLE BY EDWARDS AND MCCOMB

For discussion purpose, hereafter we refer to Edwards and McComb's theory as EM69. We use (M90; #) to represent equation number (#) in McComb's book [8]. Edwards and McComb [3] considered stationary, homogeneous, isotropic, turbulence inside a cubic box of side L . Their Liouville equation based theory uses equation for Fourier modes $u_\alpha(\mathbf{k}, t)$ of the velocity field $u_\alpha(\mathbf{x}, t)$ governed by forced Navier-Stokes equation, written as

$$\left(\frac{\partial}{\partial t} + \nu k^2 \right) u_\alpha(\mathbf{k}, t) = M_{\alpha\beta\gamma}(\mathbf{k}) \sum_{\mathbf{j}} u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{k} - \mathbf{j}, t) + f_\alpha(\mathbf{k}, t), \quad (\text{M90; 4.81}) \quad (15)$$

where f_α represents external driving force, \mathbf{k} is wavevector and $k^2 = |\mathbf{k}|^2$. The Einstein summation convention for repeated Greek indices is utilized while writing Eq. (15) and in this section. Edwards and McComb [3] theory seeks Fokker-Planck model equation for Liouville equation. The model equation contains two model parameters, namely $r(k)$ and $s(k)$, which account for contribution of nonlinear term in Eq. (15). The dynamical damping coefficient $r(k)$ is related to damping coefficient $\omega(k)$ by

$$\omega(k) = \nu k^2 + r(k). \quad (\text{M90; 6.80}) \quad (16)$$

The coefficient $s(k)$ accounts for correlation of white noise forcing in the Langevin equation for Fokker-Planck equation. Within the framework of Edwards and McComb and for stationary turbulence, $\omega(k)$ and $s(k)$ are related by

$$2\omega(k)q(k) = d(k) \quad (\text{M90; 6.84}) \quad (17)$$

where

$$d(k) = W(k) + s(k) \quad (\text{M90; 6.79}) \quad (18)$$

and $W(k)$ accounts for correlation of forcing term $f_\alpha(\mathbf{k}, t)$. The spectral density $q(k)$ is defined by

$$\left(\frac{2\pi}{L}\right)^3 \langle u_\alpha(\mathbf{k})u_\beta(-\mathbf{k}) \rangle = D_{\alpha\beta}(\mathbf{k})q(k), \quad (\text{M90; 6.85}) \quad (19)$$

where $D_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{|\mathbf{k}|^2}$. EM69 considered entropy function $S = S[q(k), \omega(k)]$ to obtain equation for $\omega(k)$ by maximizing S , corresponding to the condition

$$\frac{\delta S}{\delta \omega(k)} + \sum_j \left[\frac{\delta S}{\delta q(j)} \right] \frac{\delta q(j)}{\delta \omega(k)} = 0. \quad (\text{M90; 7.88}) \quad (20)$$

Here

$$\frac{\delta q(j)}{\delta \omega(k)} = -\frac{d(k)\delta(k-j)}{2\omega^2(k)} + \frac{1}{2\omega(j)} \frac{\delta d(j)}{\delta \omega(k)}. \quad (\text{M90; 7.89}) \quad (21)$$

and $\delta(k-j) = 1$ when $k = j$ otherwise $\delta(k-j) = 0$. Edwards and McComb realized the difficulty in obtaining the second term on the right-hand side (rhs) of Eq. (21). After neglecting the second term, an approximate equation

$$\frac{\delta q(j)}{\delta \omega(k)} = -\frac{d(k)\delta(k-j)}{2\omega^2(k)} \quad (22)$$

was used for further calculation in EM69 [8]. This Eq. (22) suggests that $\frac{\delta q(j)}{\delta \omega(k)} = 0, \forall j \neq k$. As a consequence, EM69 used following approximate equation

$$\frac{\delta S}{\delta \omega(k)} - \left[\frac{d(k)}{2\omega^2(k)} \right] \frac{\delta S}{\delta q(k)} = 0 \quad (\text{M90; 7.90}) \quad (23)$$

to obtain equation for $\omega(k)$.

We now show that the neglect of the second term on the rhs of Eq. (21) corresponds to unphysical condition. Consequently, the use of Eq. (23) to obtain $\omega(k)$ is in error. Since EM69 is proposed for stationary turbulence,

$$\left(\frac{2\pi}{L} \right)^3 \sum_{\mathbf{j}} \frac{1}{2} \langle u_{\alpha}(\mathbf{j}) u_{\alpha}(-\mathbf{j}) \rangle = \sum_{\mathbf{j}} q(j) = \text{Constant}. \quad (24)$$

and from which we can write exact equation

$$\sum_{\mathbf{j}} \frac{\delta q(j)}{\delta \omega(k)} = \frac{\delta q(k)}{\delta \omega(k)} + \sum_{\mathbf{j}, \mathbf{j} \neq \mathbf{k}} \frac{\delta q(j)}{\delta \omega(k)} = 0. \quad (25)$$

Substituting approximate Eq. (22) of EM89 into Eq. (25) and using Eq. (17), we obtain

$$\frac{d(k)}{2\omega^2(k)} = \frac{q(k)}{\omega(k)} = 0 \quad (26)$$

and which is not correct for all k . In view of this, approximation used in EM89 to obtain $\omega(k)$ corresponds to unphysical condition $q(k) = 0$ and does not comply with conservation of energy Eq. (24) for stationary turbulence where $q(k) \neq 0, \forall k$.

It should be noted that, within the framework of EM69, the unphysical behavior can be avoided if

$$\frac{\delta S}{\delta \omega(k)} = 0 \quad (27)$$

along with $q(k) = \text{constant}, \forall k$ is used instead of Eq. (20). This means that $S = S(\omega(k))$ and second term on the rhs of Eq. (20) is equal to zero and is neglected. This kind of neglect by Qian in Q83 for function $I(\eta_i)$ was considered mathematically incorrect by McComb [8]. In our view, Eq. (27) can be considered as valid equation which seeks to optimize Entropy when energy of turbulence remains constant as $q(k) = \text{constant}, \forall k$.

IV. CONCLUDING REMARKS

Within the Eulerian framework, a very few renormalized perturbation theories of turbulence are consistent with Kolmogorov spectrum [7, 8]. In this paper, we have reexamined

two such theories proposed by Qian [10] and Edwards and McComb [3] and have revealed hidden unphysical conditions in these theories. We have suggested possible modifications, Eqs. (13), (14) and (27), to these theories but have not explored their usefulness in obtaining damping coefficient consistent with Kolmogorov spectrum. This will be explored in a broader context of our future work on turbulence theory development within the framework of Kraichnan’s direct interaction approximation [9].

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